Denoising of signals and noise extraction by sparse autoregression

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Abstract: Denoising is the process of removing noise of a noise contaminated signal, noise extraction is to distill out noise. This can be done by using autoregressive (AR) filters, if the signal was generated by an AR process and if the AR coefficients are known. We introduce a sparse AR denoising/noise extraction method by using Yule-Walker (YW) equations in combination with l_1-regularization and compare the results with a 'classical' YW based denoising/noise extraction. Simulations show that the novel approach is superior compared to the classical one for short sparse AR signals. The novel approach does not require AR order estimation and may be useful for supervised or automated denoising/noise extraction of sparse AR signals.

I. Introduction

All signal recording and processing devices produce noise. Denoising or noise reduction is the process of removing noise of a noise contaminated signal, whereas noise extraction is to distill out noise. Signal denoising and noise extraction can be viewed as two sides of a coin and the related methods may be classified into three categories: 1) time-frequency analysis, 2) matrix factorization approaches, and 3) filtering techniques. Time-frequency analysis methods can be used to remove or to extract noise both in time and frequency domain, e.g. by using wavelet transformation [1] or empirical mode decomposition [2]. Matrix factorization based approaches use signal space analysis. An example is denosing by singular value decomposition [3]. The third category uses filters to remove the unwanted and to bypass the wanted signal [4]. Filtering can be done by an adaptive filter that is capable to adjust its parameters to signal's properties. A basic approach may use autoregressive (AR) filters, which requires ample knowledge or adequate estimation of the AR coefficients. Here we present a novel ansatz that uses sparse AR modelling for signal denosing and/or noise extraction.

II. Material and methods

The subsequently presented sparse AR denoising/noise extraction method relies on the Yule-Walker (YW) equations and uses l_1-regularization to estimate sparse AR coefficients.

II.I. AR modelling

The idea behind classical AR modelling of signals is to explain the present value by a linear combination of past values plus a noise or error term. Thus, an AR model of order p may be written as

$$x_t = a_1 x_{t-1} + a_2 x_{t-2} + \cdots + a_p x_{t-p} + \varepsilon_t\ , \qquad (1)$$

where x_t is the signal value at time t, a_k, $k = 1, 2, \dots, p$, are the AR coefficients, and ε_t is assumed to be white noise.

II.II. Estimation of AR coefficients

A classical approach to estimate the AR coefficients is to solve the YW equations

$$\gamma_m = \sum_{k=1}^{p} a_k\, \gamma_{m-k} + \sigma_\epsilon^2 \delta_{m,0}\ , \qquad (2)$$

where γ_m denotes the covariance of x_t with x_{t-m}, δ is the Kronecker delta. Equation (2) represents p+1 linear equations, p equations to calculate the AR coefficients and, finally, one to estimate the noise variance by using the calculated AR coefficients. Observe that the estimation of the covariances is affected by errors due to the limited signal length, thus the AR parameters are also erroneous. This affects the prediction and thus the denoising/noise extraction capabilities of an AR based denoising/noise extraction approach significantly.

II.III. AR based denoising and noise extraction

Assume that all AR coefficients are known exactly. If so, then

$$\hat{x}_t = a_1 x_{t-1} + a_2 x_{t-2} + \cdots + a_p x_{t-p} \qquad (3)$$

is not only a prediction of the true value, it may even be considered as the denoised signal value at time t, provided that the previous signal values are due an AR process of order p. Doing this for all possible time points yields a denoised signal. If the AR coefficients are inaccurate estimates, then the denoising will be poor. If the prediction is perfect, then the noise can be simply obtained by

$$\varepsilon_t = x_t - \hat{x}_t\ . \qquad (4)$$

II.IV. Sparse AR based denoising

Equivalent to estimate the AR coefficients by solving the YW equations (2) is to solve

$$\min_{\boldsymbol{a}} \tfrac{1}{2} \|\boldsymbol{\Gamma a} - \boldsymbol{\gamma}\|_2^2\ , \qquad (5)$$

where $\mathbf{\Gamma}$ is the covariance matrix, $\mathbf{a}$ is the AR coefficient vector, $\boldsymbol{\gamma}$ is the covariance vector. Note, if the true $\mathbf{a}$ is sparse, then the solution of (5) is generally not sparse [5]. By solving

$$\min_{\boldsymbol{a}} \frac{1}{2}\|\boldsymbol{\Gamma a}-\boldsymbol{\gamma}\|_2^2+\lambda\|\boldsymbol{a}\|_1 \tag{6}$$

we can obtain a sparse solution for $p_{\text{sparse AR}} \geq p_{\text{true}}$ [6]. Note that the solution depends on the l_1-regularization parameter λ.

II.V. Numerical experiments

All programmings were done by using Scilab 5.5/6.0. Sparse AR signals were generated for various orders and Gaussian white noise. Equation (5) was solved by using Levinson algorithm for classical YW based denoising. (6) was solved by using the Beck-Teboulle proximal gradient algorithm, which is also known as FISTA [7, 8].

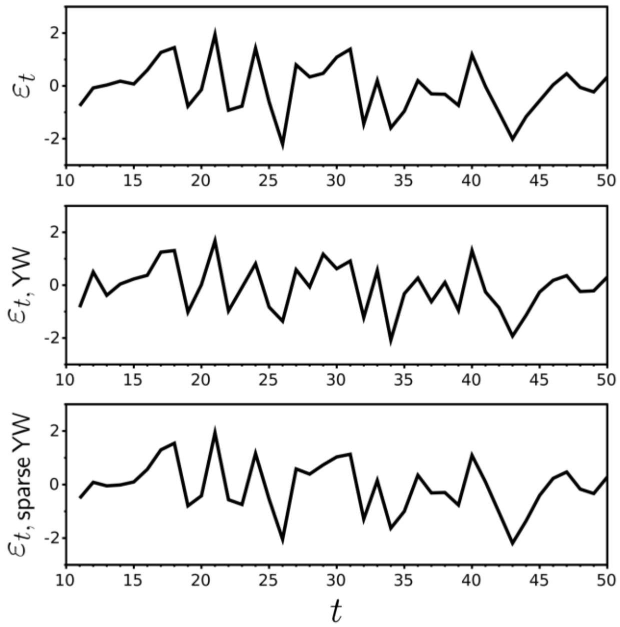

Figure 1: Noise extraction of a simulated sparse autoregressive signal. Top graph: original Gaussian white noise time course. Middle and bottom graphs show time courses obtained by classical and sparse YW based noise extraction. The parameters are listed in Table 1. Since the order of the AR process is 10, the noise reconstructions start with t = 11. Observe that the sparse noise extraction is better than the classical one.

Table 1: True and estimated AR coefficients of a sparse 10th order AR signal consisting of 50 data points. Observe that the classically YW estimated AR coefficients are not sparse. Note that the order selection for classical YW coefficient estimation is optimal.

True a_k									
-0.6	0.4	0	0	0	0	0	0	0	-0.2
YW estimated a_k									
-0.8	0.69	-0.33	0.14	0.13	-0.22	0.15	-0.04	-0.02	-0.15
Sparse YW estimated a_k									
-0.53	0.32	0	0	0.08	-0.05	0	0.02	0	-0.2

III. Results and discussion

Under the assumption that the signal to be processed is sparse, i.e. the elements of the true AR coefficient vector are mostly zero, sparse AR denoising/noise extraction is superior to classical YW based denosing/noise extraction, especially for a short signal. Fig. 1 shows a noise extraction example obtained for a sparse AR signal consisting of 50 data points. Table 1 lists true and estimated AR coefficients. Note that the classical YW approach yields non-sparse AR coefficients, whereas the novel approach does it much better. The standard deviations of the difference obtained from true and estimated AR cofficients are 0.20 for classical YW and 0.05 for sparse YW approach. The standard deviations of true minus estimated noise time courses are 0.36 for classical and 0.15 for sparse YW noise extraction. This shows that the sparse approach is better, and, in additon, we find that the sparse YW signal denoising is superior compared to the classical YW approach. This difference diminishes when much longer signals are processed. In addition, the longer the signal, the better the AR coefficient estimation – for both approaches.

If the number of the signal values is high, then both methods are, so to speak, equal. This is due to the fact that in such a case the covariance estimates are much more accurate. Note that the accuracy of the covariances influences both AR coefficient estimation procedures and thus the denoising/noise extraction quality crucially. However, the setting of the regularization parameter must be done carefully. If not, the sparse YW coefficient estimation approach can produce less good estimates, i.e. suboptimal denoising/noise extraction results. Finally we note that the introduced sparse AR estimation procedure does not require an AR order estimation process like AIC or BIC [9], but requires $p_{\text{sparse AR}} \geq p_{\text{true}}$.

IV. Conclusions

We introduced sparse AR denoising/noise extraction. First results suggest that the presented approach can be useful for supervised or automated denoising or noise extraction of a signal generated by a sparse AR process. Future work should pay attention to signal driven regularization parameter estimation and to statistical evaluation with simulated and real data.

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AUTHOR'S STATEMENT

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